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LETTER TO THE EDITOR

# Acceleration boundary for an electrically charged particle within the field of a rotating magnetic dipole

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**Abstract.** Certain aspects of the dynamics of electrically charged mass points in the vacuum field of a magnetic dipole rotating at an angular velocity perpendicular to the dipole vector are discussed. These aspects concern the so-called acceleration boundary, i.e. the limits of the spatial region in which particles such as protons and electrons, for example, are accelerated to extremely high energies. The analytical expression found for the radius of the range of influence defined in this sense for the rotating magnetic dipole in units of the light radius is found to be proportional to the $\frac{2}{3}$rd power of the magnetic dipole moment and to the $\frac{4}{3}$rd power of the angular velocity.

In this letter I wish to discuss certain aspects of the dynamics of a particle of mass $m$ and electric charge $e$ within the electromagnetic vacuum field of a magnetic dipole rotating at an angular velocity $\boldsymbol{\omega}$ perpendicular to the dipole vector $\boldsymbol{\mu}$. It is expected that some features of this configuration may become relevant for a theoretical understanding of cosmic-ray particle acceleration in pulsar magnetospheres. But it should also be made clear that the present results do not as yet claim to offer a self-consistent comprehensive theory of pulsars as cosmic-ray particle accelerators.

The problem described above has been treated recently through numerical integration of the Lorentz–Dirac equation for protons and electrons for a wide range of parameter values $\omega$ and $\mu$ [1, 2]. A typical set of parameter values is $\omega = 20\pi\ \mathrm{s}^{-1}$ and $\mu = 10^{30}\mathrm{G\ cm}^3$. For example, the trajectories as well as the energy changes of protons initially at rest at certain specified positions within the electromagnetic field of the spinning dipole have been calculated [3].

Among other findings, the existence of a 'critical surface' for protons has been discovered, dividind the space around the dipole into two regimes: an interior one, from which the protons are drawn towards the dipole, and an exterior one, from which they are accelerated to very high energies and propagated to very large distances.

It has also been found that the near-field contributions are essential for those protons which achieve extremely high energies. But outside a certain range of distance from the spinning dipole the relative importance of the near-field contributions diminishes and the pure wave field approximation becomes applicable. For the typical set of parameter values this transition has been found to take place at about 1.000 light radii distance from the dipole. The light radius is defined by $r_{\mathrm{L}} = c/\omega = \lambda/2\pi$. Furthermore it has been found that radiation reaction on protons may be neglected within the range of applicability of the pure wave field approximation.

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It is inside this range of applicability of the pure wave field approximation that the ability of the electromagnetic field to accelerate protons to very high energies and to propagate them to very large distances is found to break down within a comparatively narrow interval of initial radial distance. The topography of this 'acceleration boundary' for protons can therefore be studied by numerical integration of the equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma \boldsymbol{v}) = \frac{e}{m}\boldsymbol{E} + \frac{e}{mc}[\boldsymbol{v}, \boldsymbol{B}] \tag{1}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the appropriate electric and magnetic field vectors, respectively, representing a spherical wave of the amplitude

$$E = B - \mu / r_{\mathrm{L}}^2 r \tag{2}$$

where $r$ is the distance from the spinning dipole. This wave is linearly polarised within the equatorial plane—with the electric vector perpendicular to this plane—and circularly polarised along the axis of rotation. It is elliptically polarised in the intermediate region. $\boldsymbol{v} = (v_x, v_y, v_z)$ is the velocity vector of the proton and $\gamma = (1 - v^2/v^2)^{-1/2}$ is its Lorentz factor. Although written in a non-covariant form, the equation of motion (1) is correct as far as special relativity and classical electrodynamics are concerned and as long as radiation reaction may be neglected.

In what follows, I will discuss the 'acceleration boundary' with restriction to the equatorial plane, where the outgoing wave is linearly polarised, as was said before. Also, since this is a very localised phenomenon in terms of the radial coordinate $r$, it may be dealt with approximately using plane waves

$$\boldsymbol{E} = \boldsymbol{y}_0 E \cos[\omega(t - x/c)] \tag{3}$$

$$\boldsymbol{H} = \boldsymbol{z}_0 B \cos[\omega(t - x/c)] \tag{4}$$

instead of spherical ones. $\boldsymbol{x}_0$, $\boldsymbol{y}_0$ and $\boldsymbol{z}_0$ are unit vectors defining a cartesian coordinate system in which $\boldsymbol{x}_0$ coincides with the direction of the propagation of the electromagnetic wave.

The components of the equation of motion (1) are

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma v_x) = \omega_L v_y \cos[\omega(t - x/c)] \tag{5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma v_y) = \omega_L c(1 - v_x/c) \cos[\omega(t - x/c)] \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma v_z) = 0 \tag{7}$$

with $\omega_L = eB/mc$. An additional, though redundant, equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma c) = \omega_L v_y \cos[\omega(t - x/c)] \tag{8}$$

is obtained through multiplication of equation (1) by $\boldsymbol{v}$.

From the combination of equations (5) and (8) one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}[\gamma(1 - v_x/c)] = 0 \tag{9}$$

which means that the differential of the eigentime $d\tau = \gamma^{-1}\,dt$ equals the differential of phase, $d\tau = d(t - x/c)$. For the purposes of this letter it is sufficient to choose the following initial conditions: $x = y = z = 0$ and $\tau = 0$ for $t = 0$. The initial phase thus corresponds to maximum field strength. Furthermore $v_x = v_y = v_z = 0$ for $t = 0$. Under these specifications the eigentime is equal to the phase of the particle, $\tau = t - x/c$. Equation (6) may be integrated:

$$\gamma v_y = (\omega_L/\omega) c \sin(\omega\tau) \tag{10}$$

and inserted into equation (5):

$$\frac{d}{d\tau}(\gamma v_x) = \omega_L(\gamma v_y)\cos(\omega\tau) \tag{11}$$

which can then be integrated twice:

$$\gamma v_x / c = \tfrac{1}{4}(\omega_L/\omega)^2[1 - \cos(2\omega\tau)] \tag{12}$$

$$x/\tfrac{1}{2}\lambda = (1/8\pi)(\omega_L/\omega)^2[2\omega\tau - \sin(2\omega\tau)]. \tag{13}$$

The $x$ component of the relativistic velocity is a periodic function of the eigentime $\tau$ with a maximum value

$$\max(\gamma v_x/c) = \tfrac{1}{2}(\omega_L/\omega)^2. \tag{14}$$

The length of one period with respect to the $x$ coordinate is

$$x_P = \tfrac{1}{8}(\omega_L/\omega)^2\lambda. \tag{15}$$

As may be seen from equations (12) and (13) the $x$ component of the relativistic velocity experiences its strongest increase as a function of the $x$ coordinate at the very beginning of the trajectory, where

$$\gamma v_x/c \simeq (6^{2/3}/2)(\omega_L x/c)^{2/3}. \tag{16}$$

At $x = \frac{1}{2}x_p$ the particle passes from the region where it is accelerated with respect to the $x$ coordinate into a region where it is decelerated due to the reversal of the field vectors. Eventually the $x$ component of the relativistic velocity becomes zero at $\tau = \pi/\omega$, from where the cycle starts again.

Locally the conditions within the spherical wave field for all practical purposes are the same as in the plane wave field. Introducing the parameter

$$r_T = (e\mu/mc^2)^{1/2} \tag{17}$$

which is related to the parameters defined previously through $\omega_L/\omega = r_T^2/r_L r_0$, equation (16) may be written as

$$\gamma v_x/c \simeq (6^{2/3}/2)(r_T/r_L)^{4/3}(x/r_0)^{2/3}. \tag{18}$$

But on a larger scale the motion of the particle is modified by the decrease in the field amplitude with increasing radial distance. For example, the field amplitude is reduced by the factor $\frac{1}{2}$ within an interval of radial distance $x_D = r_0$. Obviously in the spherical wave field there is competition between the two parameters $x_D = r_0$ and

$$x_p = \tfrac{1}{4}\pi r_T^4/r_L r_0^2 \tag{19}$$

as far as their dependence on the initial radial distance $r_0$ is concerned.

At a comparatively large distance from the spinning dipole one has $x_p \ll x_D$. Then an oscillatory behaviour of $\gamma v_x/c$ is expected to be qualitatively similar to the one in the plane wave field.

At a comparatively small distance from the spinning dipole one has $x_p \gg x_D$. In this regime the particle is not expected to pass into the region of deceleration with respect to the $x$ coordinate before the field amplitude has decreased considerably. Therefore in this case $\gamma v_x/c$ will not be brought down to zero but instead will approach a certain asymptotic value (with the superposition of a small decreasing oscillating modulation).

The transition between these two regimes, i.e. the position of the 'acceleration boundary', is thus expected to occur at about $x_p \simeq x_D$, corresponding to the radial distance

$$r_B \simeq (r_T/r_L)^{1/3} r_T \tag{20}$$

which, alternatively, may be written in the form

$$r_B/r_L \simeq (e\mu/mc^2)^{2/3}(\omega/c)^{4/3}. \tag{21}$$

For the typical set of parameter values suggested at the start of this letter one finds $r_L \simeq 5\times10^8$ cm, $r_T \simeq 5\times10^{11}$ cm and consequently $r_B \simeq 10^4\, r_L$.

## References


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